Morphisms to $\mathbb{P}^n$ (Har II 7)

Let A be a ring, $S = A[x_0, \dots, x_n]$, and X a scheme over A. Suppose there is a morphism

$$\varphi : X \longrightarrow \mathbb{P}^n_A .$$

Let $\mathcal{L} = \varphi^*(\mathcal{O}(1))$. Recall that the stalk $\mathcal{L}_x = \mathcal{O}(1)_{\varphi(x)}$, so since $\mathcal{O}(1)$ is globally generated by $x_0, \dots, x_n$, $\mathcal{L}$ will also be globally generate by the pullbacks of the sections: $s_i = \varphi^*(x_i) \in \Gamma(X, \mathcal{L})$.

Conversely, we have the following:

Theorem: Let X be as above, and $\mathcal{L}$ an invertible sheaf on X. If $\mathcal{L}$ is globally generated by $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$, there is a unique A-morphism $\varphi : X \to \mathbb{P}^n_A$ s.t. $\mathcal{L} \cong \varphi^* \mathcal{O}(1)$ and $s_i = \varphi^*(x_i)$.

Pf: Define $X_i = \{ P \in X \mid (s_i)_P \notin m_P \mathcal{L}_P \}$, an open subset of X. Since $\mathcal{L}_P$ is free of rank one, it is generated by any element not in $m_P \mathcal{L}_P$, so s_i generates the stalk $\mathcal{L}_P$ at each $P \in X_i$.

For each P, the $(s_i)_P$ generate $\mathcal{L}_P$. Thus, they're not all contained in $m_P \mathcal{L}_P$, so the X_i cover X.

Let $\{U_i\}$ be the standard open cover of $\mathbb{P}^n$, where $U_i = \{x_i \neq 0\}$, so that

$$U_i \cong \operatorname{Spec} A[x_0/x_i, x_1/x_i, \ldots, x_n/x_i]$$

Define a ring homomorphism

$$R_i = A[x_0/x_i, \ldots, x_n/x_i] \to \Gamma(X_i, \mathcal{O}_{X_i}) \quad \text{by}$$

$$x_j/x_i \longmapsto s_j/s_i .$$

$s_j, s_i \in \Gamma(X, \mathcal{L})$, so why does this make sense?
For $p \in X_i$, $(s_i)_p$ generates $\mathcal{L}_p$, so there's a unique $r_p \in \mathcal{O}_p$ s.t. $(s_j)_p = r_p (s_i)_p$.

Thus, there's a unique $r \in \mathcal{O}_X(X_i)$ s.t. $s_j = r s_i$.
Set $s_j/s_i = r$.

The maps $R_i \to \Gamma(X_i, \mathcal{O}_{X_i})$ uniquely determine morphisms $X_i \longrightarrow \operatorname{Spec} R_i$ (see e.g. Hartshorne exercise 2.4). These morphisms glue together (check!), giving us an A-morphism $\varphi: X \to \mathbb{P}^n_A$.

$$\begin{aligned}
\text{On each } X_i,\quad \varphi^* \mathcal{O}(1)(X_i) &= \underset{\substack{\uparrow \\ \text{sections of } \mathcal{O}(1)}}{R_i x_i} \otimes \Gamma(X_i, \mathcal{O}_{X_i}) \\
&= \Gamma(X_i, \mathcal{O}_{X_i})\, s_i \\
&= \mathcal{L}(X_i).
\end{aligned}$$

Since $\mathcal{L}$ is locally free and has the same trivializations as $\varphi^*\mathcal{O}(1)$, we have $\mathcal{L} \cong \varphi^*\mathcal{O}(1)$ and $s_i = \varphi^*(x_i)$. $\square$

What is happening here? We are sending the $s_i \neq 0$ patch to the $x_i \neq 0$ patch. In the $A = k$ case, why can't we take a k-point P and send it to $[s_0(P) : \dots : s_n(P)]$? Because even though $\mathcal{L}|_{X_i} \cong \mathcal{O}_{X_i}$, the value of $s_i(P) \in k(P)$ depends on the trivialization, i.e. the choice of isomorphism. However, the ratio s_i/s_j will be consistent.

Example: $\mathcal{O}(3)$ on $\mathbb{P}^1$ is globally generated by x^3, x^2y, xy^2, y^3, which define a morphism

$$\begin{aligned} \mathbb{P}^1 &\longrightarrow \mathbb{P}^3 \\ [x:y] &\longmapsto [x^3 : x^2y : xy^2 : y^3] \end{aligned}$$

(twisted cubic)

This is an example of a Veronese embedding. More generally, let $X = \mathbb{P}^n_k$, and $\mathcal{L} = \mathcal{O}_X(d)$, which is globally generated for $d \geq 1$ by $\binom{d+n}{n}$ sections (e.g. all deg d monomials).

Thus, this determines a morphism

$$\mathbb{P}^n_k \longrightarrow \mathbb{P}^{\binom{d+n}{n}-1}$$

called the d^{th} Veronese embedding.

Since the only invertible sheaves on $\mathbb{P}^n_k$ are $\mathcal{O}(d)$ for $d \in \mathbb{Z}$, all of the maps between projective spaces come from subsets of global sections of some $\mathcal{O}(d)$.

In particular, one can show that the automorphisms of $\mathbb{P}^n$ correspond to k-bases of $\Gamma(\mathbb{P}^n, \mathcal{O}(1))$ up to scaling. i.e. the automorphism group is $PGL(n)$.

What happens if we choose a set of global sections which don't generate $\mathcal{L}$?

Ex: $X = \mathbb{P}^2_k$, and take $xy, yz, xz \in \Gamma(X, \mathcal{O}_X(2))$.
These sections don't generate $\mathcal{O}_X(2)$ at each stalk:

Set $P = [1:0:0]$. Then $xy, yz, xz \in m_P \mathcal{O}(2)_P$, since the maximal ideal is generated by $\frac{y}{x}, \frac{z}{x}$. Thus, these sections can't generate the stalk.

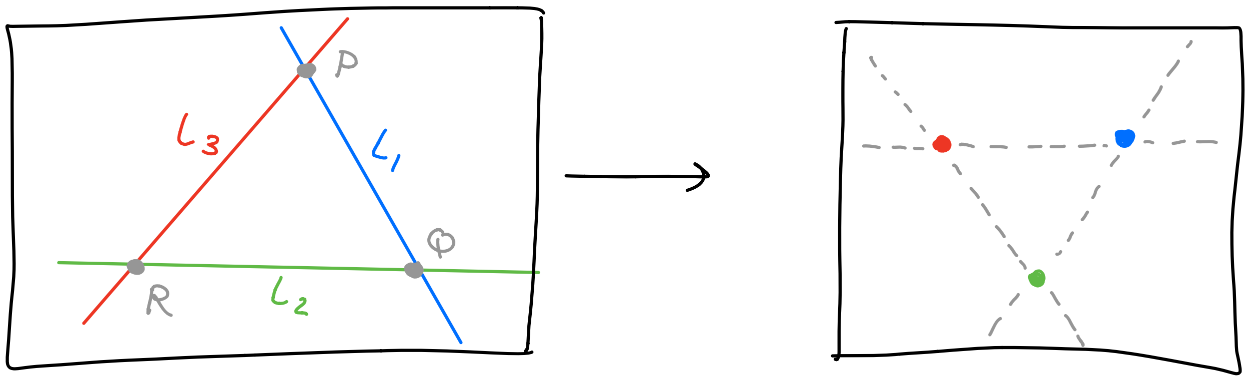
If we try to construct a map anyway, we get

$$[x:y:z] \longmapsto [xy:xz:yz].$$

Problem: $\underset{P}{[1:0:0]}, \underset{Q}{[0:1:0]}, \underset{R}{[0:0:1]} \longmapsto [0:0:0]$, so it's not defined at these points.

However, it's defined at all other points, so it defines a map $\mathbb{P}^2 - \{P, Q, R\} \longrightarrow \mathbb{P}^2$. (Called a Cremona transformation.)

What's the image?



$L_1 = \{[* : * : 0]\} \longmapsto [1:0:0]$
$L_2 = \{[0 : * : *]\} \longmapsto [0:0:1]$
$L_3 = \{[* : 0 : *]\} \longmapsto [0:1:0]$

This works more generally: If X is a scheme over A, $\mathcal{L}$ an invertible sheaf, and $s_0, \dots, s_n$ any set of global sections, let $U \subseteq X$ be the (possibly empty) open set over which the s_i generate $\mathcal{L}$.

Then $\mathcal{L}|_U$ and the $s_i|_U$ give a morphism $U \to \mathbb{P}^n_A$.

We'll come back to the concrete setting of projective varieties soon when we discuss linear systems.

Closed immersions

Under what conditions do $\mathcal{L}$ and $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$ determine a closed immersion?

Prop: Let $\varphi: X \to \mathbb{P}^n_A$ be a morphism of A-schemes, corresponding to an invertible sheaf $\mathcal{L}$ on X and sections $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$. Then φ is a closed immersion iff

1. each open set X_i (as defined above) is affine, and

2. for each i, the map $A[y_0, \dots, y_n] \to \Gamma(X_i, \mathcal{O}_{X_i})$ defined $y_j \mapsto s_j/s_i$ is surjective.

Pf: If φ is a closed immersion, then $X_i = X \cap U_i$ is a closed subscheme of U_i, so it's affine, and the corr. map of rings is surjective.

Conversely, if 1 and 2 are satisfied, then each X_i is a closed subscheme of U_i. Since the X_i cover X, X must be a closed subscheme of $\mathbb{P}^n$. $\square$

If we are over an alg. closed field, we can give a criterion on stalks. (For proof, see Har 7.3)

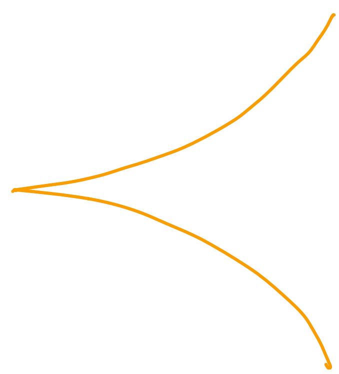
Prop: Let $k = \bar{k}$, X a projective scheme over k, and

$\varphi : X \to \mathbb{P}^n_k$ a morphism corresponding to $\mathcal{L}$ and $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$. Let $V \subseteq \Gamma(X, \mathcal{L})$ be the subspace spanned by the s_i. φ is a closed immersion if and only if

1. for $P, Q \in X$ distinct closed points $\exists$ $s \in V$ s.t. $s \in m_P \mathcal{L}_P$ but $s \notin m_Q \mathcal{L}_Q$ or vice versa (i.e. it vanishes at one pt but not the other), and

2. for each closed $P \in X$, $\{ s \in V \mid s_P \in m_P \mathcal{L}_P \}$ spans $m_P \mathcal{L}_P / m_P^2 \mathcal{L}_P$ (i.e. V "separates tangent vectors").

Ex: Consider $\mathcal{O}(3)$ on $\mathbb{P}^1$, along w/ sections st^2, t^3, s^3. The image of $\varphi : [s:t] \to [st^2 : t^3 : s^3]$ in $\mathbb{P}^2$ is the curve defined $x^3 = y^2 z$.



If $P = [a:1]$, $Q = [b:1]$, then $st^2 - at^3$ vanishes at P but not Q. Equivalently, $st^2 - at^3 \in (s - at)$ in the stalk.

If $P = [1:0]$, t^3 vanishes at P but not Q.

However, at $P = [1:0]$, the set of sections of V in $m_P \mathcal{O}(3)_P$ is generated by st^2 and t^3, whereas

$$m_P \mathcal{O}(3)_P = (t) \left(k[s,t](3)_{(t)} \right)_0$$

is generated by t, so $st^2, t^3 \in m_p^2 \mathcal{O}(3)_p$. Thus, V does not separate tangent vectors.

Very ampleness + ampleness

Let X be a scheme of finite type over a Noetherian ring A, and $\mathcal{L}$ an invertible sheaf on X.

Def: $\mathcal{L}$ is <u>very ample</u> if it admits a set of global sections $s_0, \dots, s_n$ such that the corresponding morphism $X \to \mathbb{P}^n_A$ is an immersion (open subset of a closed embedding).

(If X is proper$/k$, it's a closed immersion)

$\mathcal{L}$ is <u>ample</u> if $\mathcal{L}^{\otimes n}$ is very ample for some $n > 0$.

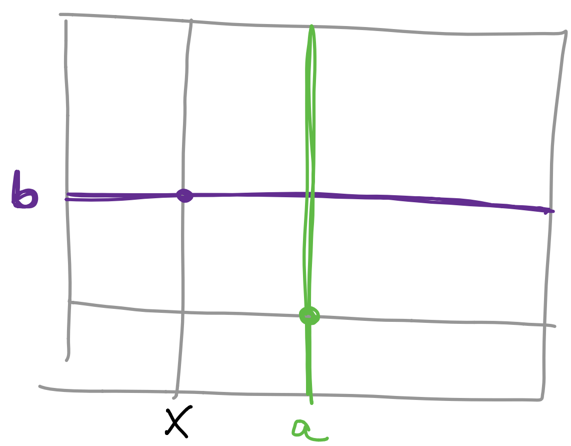
There are several equivalent characterizations of ampleness, some involving cohomology. Here is one we can state now:

Prop: Let $\mathcal{L}$ be an invertible sheaf on X (finite type over Noetherian A). Then $\mathcal{L}$ is ample iff for every coherent $\mathcal{F}$ on X, there's an integer $n_0 > 0$ s.t. for $n \geq n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated.

This characterization can be given as the definition of ample in a more general setting (e.g. it doesn't require finite type).

Ex: On $\mathbb{P}^n_k$, we saw that $\mathcal{O}(d)$, $d \geq 1$ determines the d^{th} Veronese embedding. Thus, $\mathcal{O}(d)$ is ample and very ample. If $d \leq 0$, then $\mathcal{O}(d)^n = \mathcal{O}(dn)$ which is not globally generated. Thus, $\mathcal{O}(d)$ is ample $\Leftrightarrow$ $d>0$.

Ex: Let $Q \subseteq \mathbb{P}^3_k$ be the nonsingular quadric surface. We saw that $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, and $\operatorname{Pic} Q \cong \mathbb{Z} \oplus \mathbb{Z}$. Let $\mathcal{L}$ be an invertible sheaf on Q, of type (a,b). Then if $a, b > 0$, $\mathcal{L}$ corresponds to the product of Veronese embedding composed w/ a Segre embedding



$$\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \longrightarrow \mathbb{P}^n,$$

so it is very ample. However, if a or b is ≤ 0, we can restrict $\mathcal{L}$ to the corresponding fiber X, and get $\mathcal{L}|_X \cong \mathcal{O}_X(b)$ (WLOG assume $b \leq 0$), which is not globally generated. Thus, $\mathcal{L}$ is not ample.

Ex: Let $X = \operatorname{Spec} A$ an affine scheme. Every coherent sheaf on X is of the form $\tilde{M}$, where M is an A-module. Thus, every coherent sheaf is globally generated, so every invertible sheaf is ample.